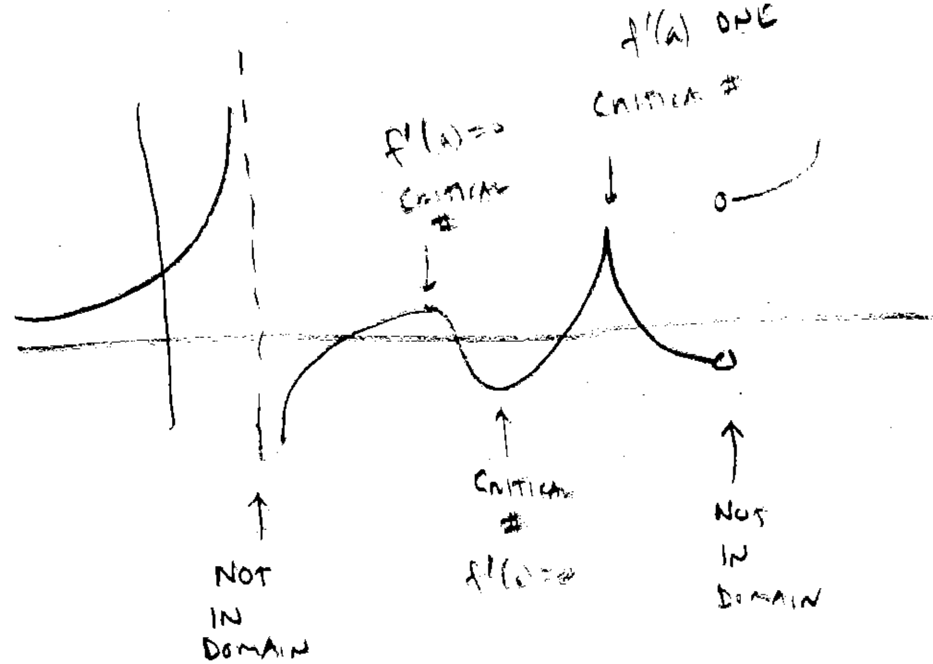
4.1: Critical Points and Absolute Max/Min

Given $y = f(x)$.
The first questions we always ask:

1. What is the domain?
   (What inputs are allowed?)

2. What are the "critical numbers"?
   A **critical number** is a number $x = a$ that is <u>in the domain</u> and either
   (a) $f'(a) = 0$, or
   (b) $f'(a)$ does not exist.



$\frac{1}{x-2} \rightarrow$ DOMAIN $x \neq 2$

$\sqrt{x+3} \rightarrow$ DOMAIN $x \geq -3$

$\ln(x) \rightarrow$ DOMAIN $x > 0$.

Example:

$$f(x) = 4x + \frac{1}{x} \quad = 4x + x^{-1}$$

a) What is the domain? $\boxed{x \neq 0}$

b) What are the critical numbers?

$$f'(x) = 4 - x^{-2} = 4 - \frac{1}{x^2}$$

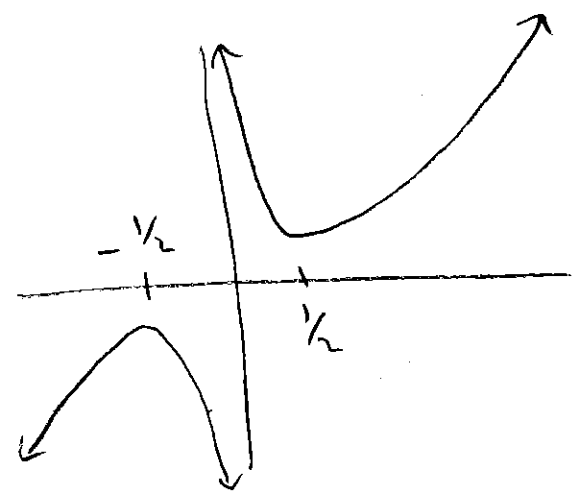
NOTE: $f'(x)$ DNE AT $x=0$, but $x=0$ IS NOT IN DOMAIN. (NOT A "CRITICAL NUMBER")

$$4 - \frac{1}{x^2} \stackrel{?}{=} 0$$

$$4x^2 - 1 = 0$$

$$x^2 = 1/4$$

$$\boxed{x = \pm 1/2}$$



Example (from homework):

$$y = x^3 + 3x^2 - 72x$$

a) What is the domain? $\boxed{\text{ALL REAL \#'s}}$

b) What are the critical numbers?

$$y' = 3x^2 + 6x - 72 \stackrel{?}{=} 0$$

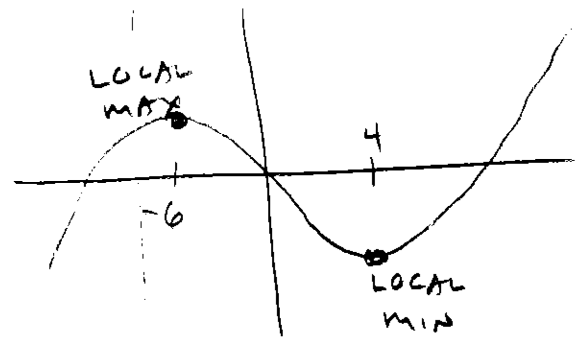
$$3(x^2 + 2x - 24) \stackrel{?}{=} 0$$

$$3(x+6)(x-4) \stackrel{?}{=} 0$$

$$\boxed{\begin{array}{l} x = -6 \\ x = 4 \end{array}}$$

CRITICAL NUMBERS

ASIDE



Example:

$$g(x) = 3x - x^{1/3} \quad = 3x - \sqrt[3]{x}$$

ODD ROOT

a) What is the domain? $\boxed{\text{ALL REAL NUMBERS}}$

b) What are the critical numbers?

$$g'(x) = 3 - \frac{1}{3}x^{-2/3} = 3 - \frac{1}{3x^{2/3}}$$

$g'(x)$ DNE AT $x=0$ AND $x=0$ IS IN THE DOMAIN.

$\boxed{x=0 \text{ IS A CRITICAL NUMBER.}}$ VERTICAL TANGENT

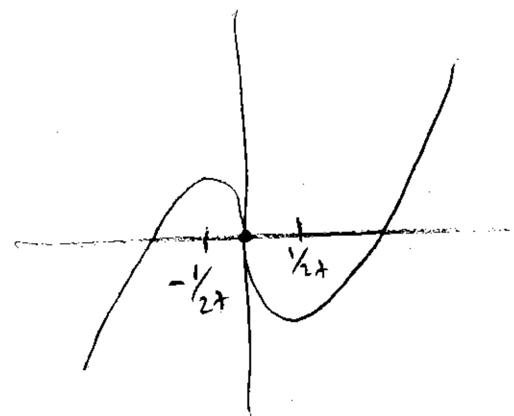
$$3 - \frac{1}{3x^{2/3}} \stackrel{?}{=} 0$$

$$9x^{2/3} - 1 = 0$$

$$x^{2/3} = 1/9$$

$$\boxed{x = \pm\left(1/9\right)^{3/2} = \pm\frac{1}{27}}$$

CRITICAL #'s



Absolute Max/Min

The **absolute max** (or **global** max) is the highest *y*-value on the interval.

The **absolute min** (or **global** min) is the lowest *y*-value on the interval.

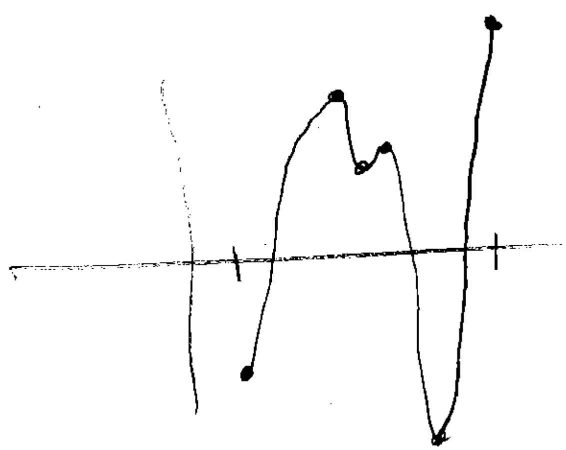
Procedure to find absolute max/min:

1. Find critical numbers.
2. Plug endpoints and critical numbers into the function.

<u>Big, key, awesome observation</u>:

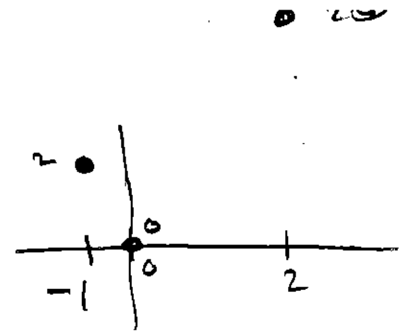
(*Extreme Value Theorem*)

The absolute max/min always occur at critical numbers or endpoints!



Example (like HW):

Find the abs. max and min of $f(x) = x^3 + 3x^2$ on $[-1,2]$.

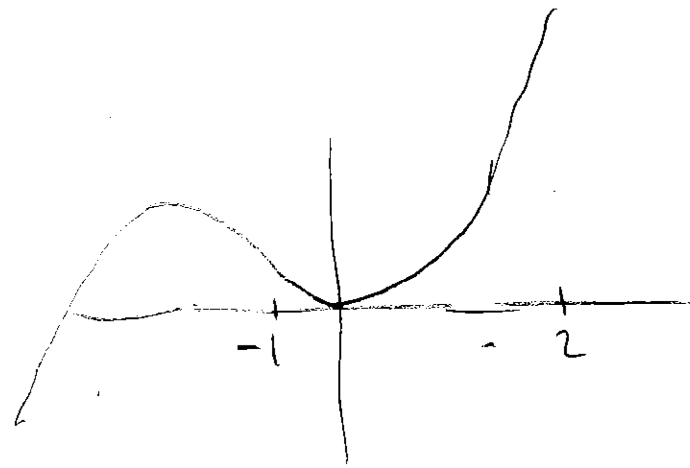


$$f'(x) = 3x^2 + 6x \stackrel{?}{=} 0$$

$$3x(x+2) = 0$$

$$x = 0 \qquad x = -2$$

ONLY CRITICAL # IN THIS DOMAIN



$$f(-1) = (-1)^3 + 3(-1)^2 = -1 + 3 = 2$$

$$f(0) = 0$$

$$f(2) = (2)^3 + 3(2)^2 = 8 + 12 = 20$$

$$\boxed{\text{ABS MAX} = 20}$$

$$\boxed{\text{ABS. MIN} = 0}$$

Small Note:
The **value** of a function, $y = f(x)$, is the output y-value. A question asking for the absolute max of a function is asking for the **y-value**. (The x-value is the location where the max *occurs*)

Example: $\boxed{\text{DOMAIN } x > 0}$

Find the abs. max and min of $f(x) = x\ln(x)$ on $[1, e]$.

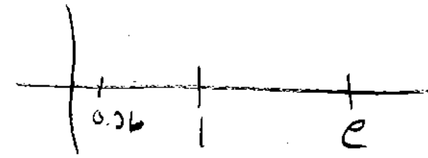
$$f'(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x)$$

$$1 + \ln(x) \overset{?}{=} 0$$

$$\ln(x) = -1$$

$$x = e^{-1} \approx 0.367879$$

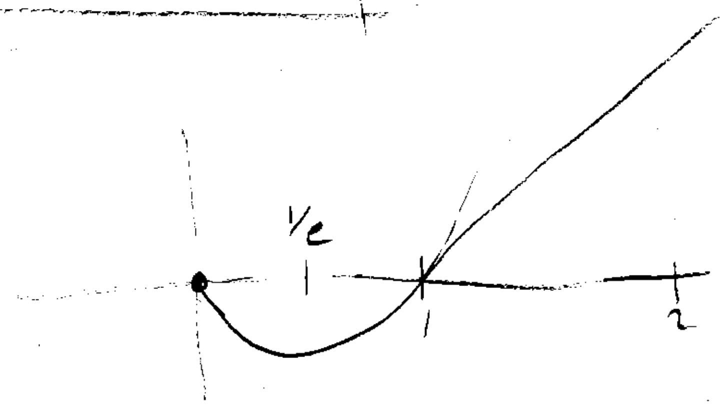
NOT IN GIVEN DOMAIN!



$f(1) = 1 \cdot \ln(1) = 0$

$f(e) = e\ln(e) = e$

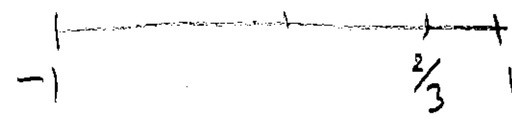
$\boxed{\begin{array}{l} \text{ABS. MAX} = e \\ \text{ABS. MIN} = 0 \end{array}}$ occurs at $x=1$ / occurs at $x=e$



Example:

DOMAIN $x \le 1$

Find the abs. max and min of $f(x) = x\sqrt{1-x}$ on $[-1,1]$.



$$f'(x) = \sqrt{1-x} + x\,\frac{-1}{2\sqrt{1-x}} \stackrel{?}{=} 0$$

$$1 - x - \tfrac{1}{2}x = 0$$

$$1 - \tfrac{3}{2}x = 0$$

$$1 = \tfrac{3}{2}x$$

$$\boxed{x = 2/3}$$

Critical pts

$f'(x)$ DNE AT $\boxed{x=1}$ ← IN DOMAIN.

$f(-1) = (-1)\sqrt{1-(-1)} = -\sqrt{2} \approx -1.414$

$f(2/3) = \frac{2}{3}\sqrt{1-2/3} = \frac{2}{3}\sqrt{1/3}$

$= \frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{3}$

≈ 0.3849

$f(1) = (1)\sqrt{1-1} = 0$

ABS. MAX $= \frac{2}{3\sqrt{3}}$

ABS MIN $= -\sqrt{2}$